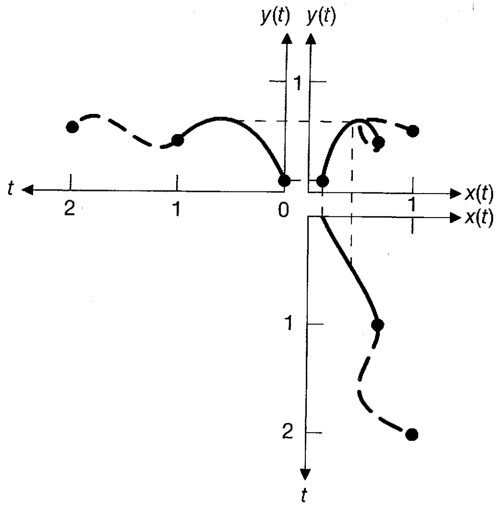

Fig. 11.7 Two joined 2D parametric curve segments and their defining polynomials. The dotted lines between the (x,y) plot and the x(t) and y(t) plots show the correspondence between the points on the (x,y) curve and the defining cubic polynomials. The x(t) and y(t) plots for the second segment have been translated to begin at $t = 1$, rather than at $t = 0$, to show the continuity of the curves at their join point.

If two curve segments join together, the curve has G^0 *geometric continuity*. If the directions (but not necessarily the magnitudes) of the two segments' tangent vectors are equal at a join point, the curve has G^1 geometric continuity. In computer-aided design of objects, G^1 continuity between curve segments is often required. G^1 continuity means that the geometric slopes of the segments are equal at the join point. For two tangent vectors TV_1 and TV_2 to have the same direction, it is necessary that one be a scalar multiple of the other: $TV_1 = k \cdot TV_2$, with $k > 0$ [BARS88].

If the tangent vectors of two cubic curve segments are equal (i. e., their directions *and* magnitudes are equal) at the segments' join point, the curve has first-degree continuity in the parameter t, or *parametric continuity*, and is said to be C^1 continuous. If the direction and magnitude of $d^n/dt^n[Q(t)]$ through the nth derivative are equal at the join point, the curve is called C^n *continuous*. Figure 11.8 shows curves with three different degrees of continuity. Note that a parametric curve segment is itself everywhere continuous; the continuity of concern here is at the join points.

The tangent vector $Q'(t)$ is the velocity of a point on the curve with respect to the parameter t. Similarly, the second derivative of $Q(t)$ is the acceleration. If a camera is moving along a parametric cubic curve in equal time steps and records a picture after each step, the tangent vector gives the velocity of the camera along the curve. The camera velocity and acceleration at join points should be continuous, to avoid jerky movements in the resulting animation sequence. It is this continuity of acceleration across the join point in Fig. 11.8 that makes the C^2 curve continue farther to the right than the C^1 curve, before bending around to the endpoint.

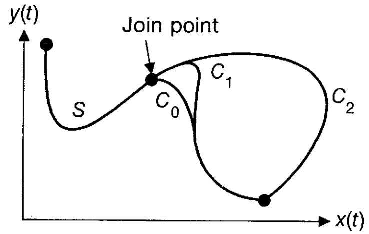

Fig. 11.8 Curve segment S joined to segments C_0, C_1, and C_2 with the 0, 1, and 2 degrees of parametric continuity, respectively. The visual distinction between C_1 and C_2 is slight at the join, but obvious away from the join.

In general, C^1 continuity implies G^1, but the converse is generally not true. That is, G^1 continuity is generally less restrictive than is C^1, so curves can be G^1 but not necessarily C^1. However, join points with G^1 continuity will appear just as smooth as those with C^1 continuity, as seen in Fig. 11.9.

There is a special case in which C^1 continuity does *not* imply G^1 continuity: When both segments' tangent vectors are $[0 \quad 0 \quad 0]$ at the join point. In this case, the tangent vectors are indeed equal, but their directions can be different (Fig. 11.10). Figure 11.11 shows this concept in another way. Think again of a camera moving along the curve; the camera velocity slows down to zero at the join point, the camera changes direction while its velocity is zero, and the camera accelerates in the new direction.

The plot of a parametric curve is distinctly different from the plot of an ordinary function, in which the independent variable is plotted on the x axis and the dependent variable is plotted on the y axis. In parametric curve plots, the independent variable t is

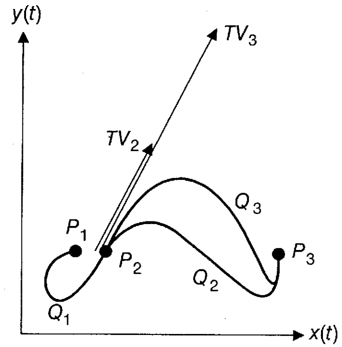

Fig. 11.9 Curve segments Q_1, Q_2, and Q_3 joint at the point P_2 and are identical except for their tangent vectors at P. Q_1 and Q_2 have equal tangent vectors, and hence are both G^1 and C^1 continuous at P_2. Q_1 and Q_3 have tangent vectors in the same direction, but Q_3 has twice the magnitude, so they are only G^1 continuous at P_2. The larger tangent vector of Q_3 means that the curve is pulled more in the tangent-vector direction before heading toward P_3. Vector TV_2 is the tangent vector for Q_2, TV_3 is that for Q_3.

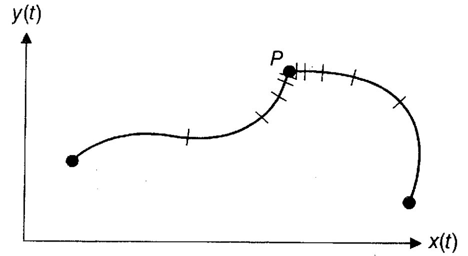

Fig. 11.10 The one case for which C^1 continuity does not imply G^1 continuity: the tangent vector (i.e., the parametric velocity along the curve) is zero at the join joint P. Each tick mark shows the distance moved along the curve in equal time intervals. As the curve approaches P, the velocity goes to zero, then increases past P.

never plotted at all. This means that we cannot determine, just by looking at a parametric curve plot, the tangent vector to the curve. It is possible to determine the direction of the vector, but not its magnitude. This can be seen as follows: if $\gamma(t)$, $0 \le t \le 1$ is a parametric curve, its tangent vector at time 0 is $\gamma'(0)$. If we let $\eta(t) = \gamma(2t)$, $0 \le t \le \frac{1}{2}$, then the parametric plots of γ and η are identical. On the other hand, $\eta'(0) = 2\,\gamma'(0)$. Thus, two curves that have identical plots can have different tangent vectors. This is the motivation for the definition of geometric continuity: For two curves to join smoothly, we require only that their tangent-vector directions match, not that their magnitudes match.

A curve segment $Q(t)$ is defined by constraints on endpoints, tangent vectors, and continuity between curve segments. Each cubic polynomial of Eq. (11.5) has four coefficients, so four constraints will be needed, allowing us to formulate four equations in the four unknowns, then solving for the unknowns. The three major types of curves discussed in this section are *Hermite*, defined by two endpoints and two endpoint tangent vectors; *Bézier*, defined by two endpoints and two other points that control the endpoint tangent vectors; and several kinds of *splines*, each defined by four control points. The

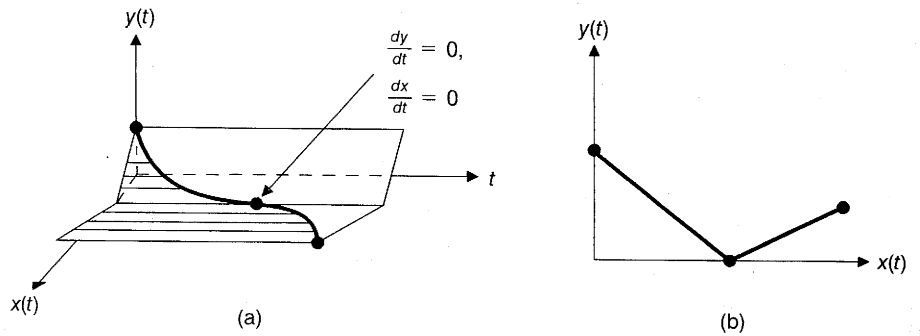

Fig. 11.11 (a) View of a 2D parametric cubic curve in 3D (x, y, t) space, and (b) the curve in 2D. At the join, the velocity of both parametrics is zero; that is, $dy/dt = 0$ and $dx/dt = 0$. You can see this by noting that, at the join, the curve is parallel to the t axis, so there is no change in either x or y. Yet at the join point, the parametrics are C^1 continuous, but are not G^1 continuous.

splines have C^1 and C^2 continuity at the join points and come close to their control points, but generally do not interpolate the points. The types of splines are uniform B-splines, nonuniform B-splines, and β-splines.

To see how the coefficients of Eq. (11.5) can depend on four constraints, we recall that a parametric cubic curve is defined by $Q(t) = T \cdot C$. We rewrite the coefficient matrix as $C = M \cdot G$, where M is a 4×4 *basis matrix*, and G is a four-element column vector of geometric constraints, called the *geometry vector*. The geometric constraints are just the conditions, such as endpoints or tangent vectors, that define the curve. We use G_x to refer to the column vector of just the x components of the geometry vector. G_y and G_z have similar definitions. M or G, or both M and G, differ for each type of curve.

The elements of M and G are constants, so the product $T \cdot M \cdot G$ is just three cubic polynomials in t. Expanding the product $Q(t) = T \cdot M \cdot G$ gives

$$Q(t) = [x(t) \quad y(t) \quad z(t)] = [t^3 \quad t^2 \quad t \quad 1] \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \cdot \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}. \quad (11.9)$$

Multiplying out just $x(t) = T \cdot M \cdot G_x$ gives

$$x(t) = (t^3 m_{11} + t^2 m_{21} + t\, m_{31} + m_{41}) g_{1x} + (t^3 m_{12} + t^2 m_{22} + t\, m_{32} + m_{42}) g_{2x} + (t^3 m_{13} + t^2 m_{23} + t\, m_{33} + m_{43}) g_{3x} + (t^3 m_{14} + t^2 m_{24} + t\, m_{34} + m_{44}) g_{4x}. \quad (11.10)$$

Equation (11.10) emphasizes that the curve is a weighted sum of the elements of the geometry matrix. The weights are each cubic polynomials of t, and are called *blending functions*. The blending functions B are given by $B = T \cdot M$. Notice the similarity to a piecewise linear approximation, for which only two geometric constraints (the endpoints of the line) are needed, so each curve segment is a straight line defined by the endpoints G_1 and G_2:

$$\begin{aligned} x(t) &= g_{1x}\,(1 - t) + g_{2x}\,(t), \\ y(t) &= g_{1y}\,(1 - t) + g_{2y}\,(t), \\ z(t) &= g_{1z}\,(1 - t) + g_{2z}\,(t). \end{aligned} \quad (11.11)$$

Parametric cubics are really just a generalization of straight-line approximations.

To see how to calculate the basis matrix M, we turn now to specific forms of parametric cubic curves.

11.2.1 Hermite Curves

The Hermite form (named for the mathematician) of the cubic polynomial curve segment is determined by constraints on the endpoints P_1 and P_4 and tangent vectors at the endpoints R_1 and R_4. (The indices 1 and 4 are used, rather than 1 and 2, for consistency with later sections, where intermediate points P_2 and P_3 will be used instead of tangent vectors to define the curve.)

To find the *Hermite basis matrix* M_H, which relates the *Hermite geometry vector* G_H to the polynomial coefficients, we write four equations, one for each of the constraints, in the four unknown polynomial coefficients, and then solve for the unknowns.