

Generating Random Variates I

Holger Füßler

Course overview

1. Introduction

2. Building block: RNG

**3. Building block:
Generating random variates I
and modeling examples**

**4. Building block:
Generating random variates II
and modeling examples**

**5. Algorithmics:
Management of events**

6. NS-2: Introduction

7. NS-2: Fixed networks

8. NS-2: Wireless networks

9. Output analysis: single system

**10. Output analysis: comparing
different configuration**

11. Omnet++ / OPNET

12. Simulation lifecycle, summary

Structure of this lecture

» Part I: Modeling examples w.r.t. $U(0,1)$

- Random graphs (topologies)
- Constant bit rate data traffic with jitter
- Randomness as protocol element
 - Example: multiaccess communication

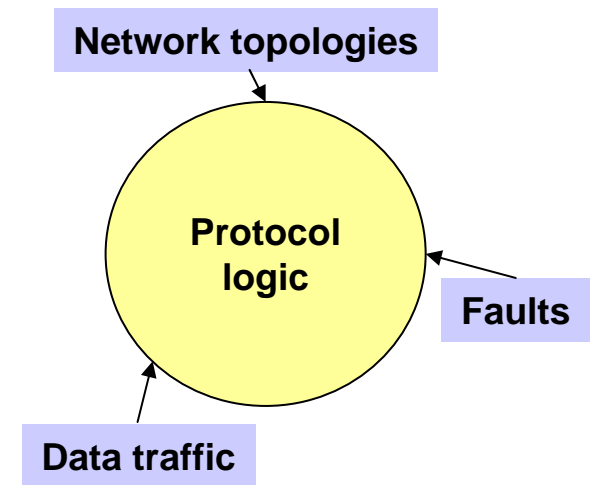
» Part II: Generation of other random variates for modeling network elements

- Inverse transform method
- Example: exponential distribution
- Inverse transform method for discrete random variates
- Generalized inverse transform

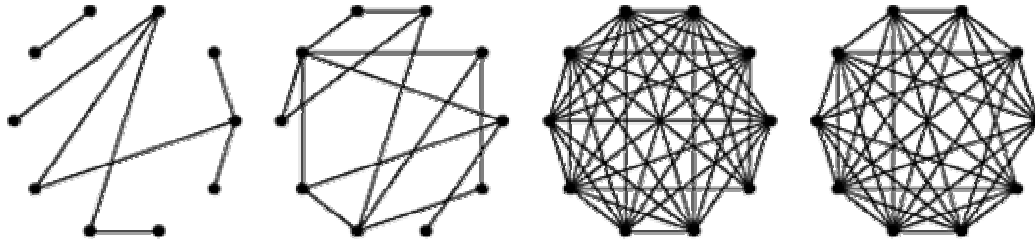
» Part III: Modeling examples w.r.t. exponential distribution

- Random direction mobility
- On/off sources for generating bursty traffic

Random effects:
'how to' and 'what'



I Random graphs I



Eric W. Weisstein. "Random Graph." From [MathWorld](http://mathworld.wolfram.com/RandomGraph.html)--A Wolfram Web Resource.
<http://mathworld.wolfram.com/RandomGraph.html>

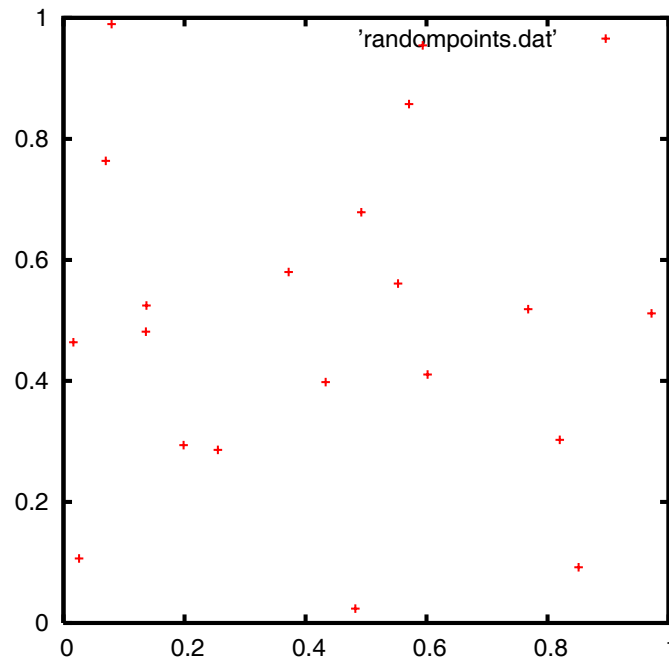
For each pair of vertices (u,v) :

- » There is an edge between u and v with probability $a \in [0,1]$.
- » Required: sampling from Bernoulli random variable; mass function:

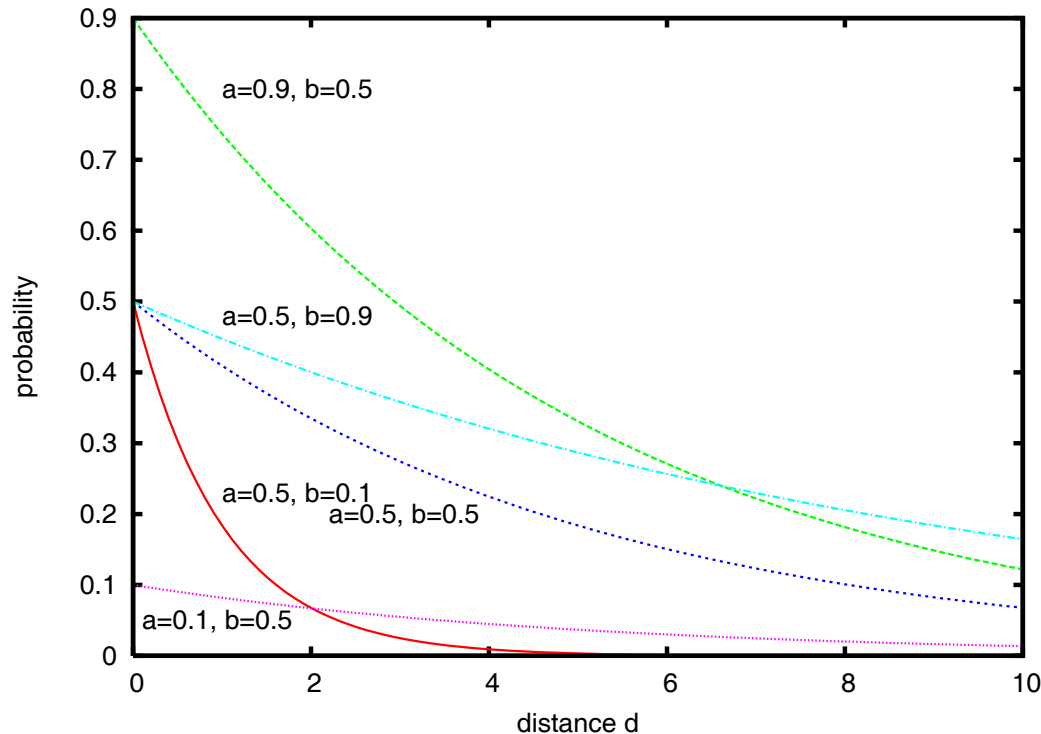
$$p(x) = \begin{cases} 1 - a & \text{if } x = 0 \\ a & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

I Random graphs II

- » **Select vertices by sampling from a 2-dimensional uniform distribution**
 - **Since dimensions should be independent, one can simply sample x and y coordinates from a 1-dimensional uniform distribution**



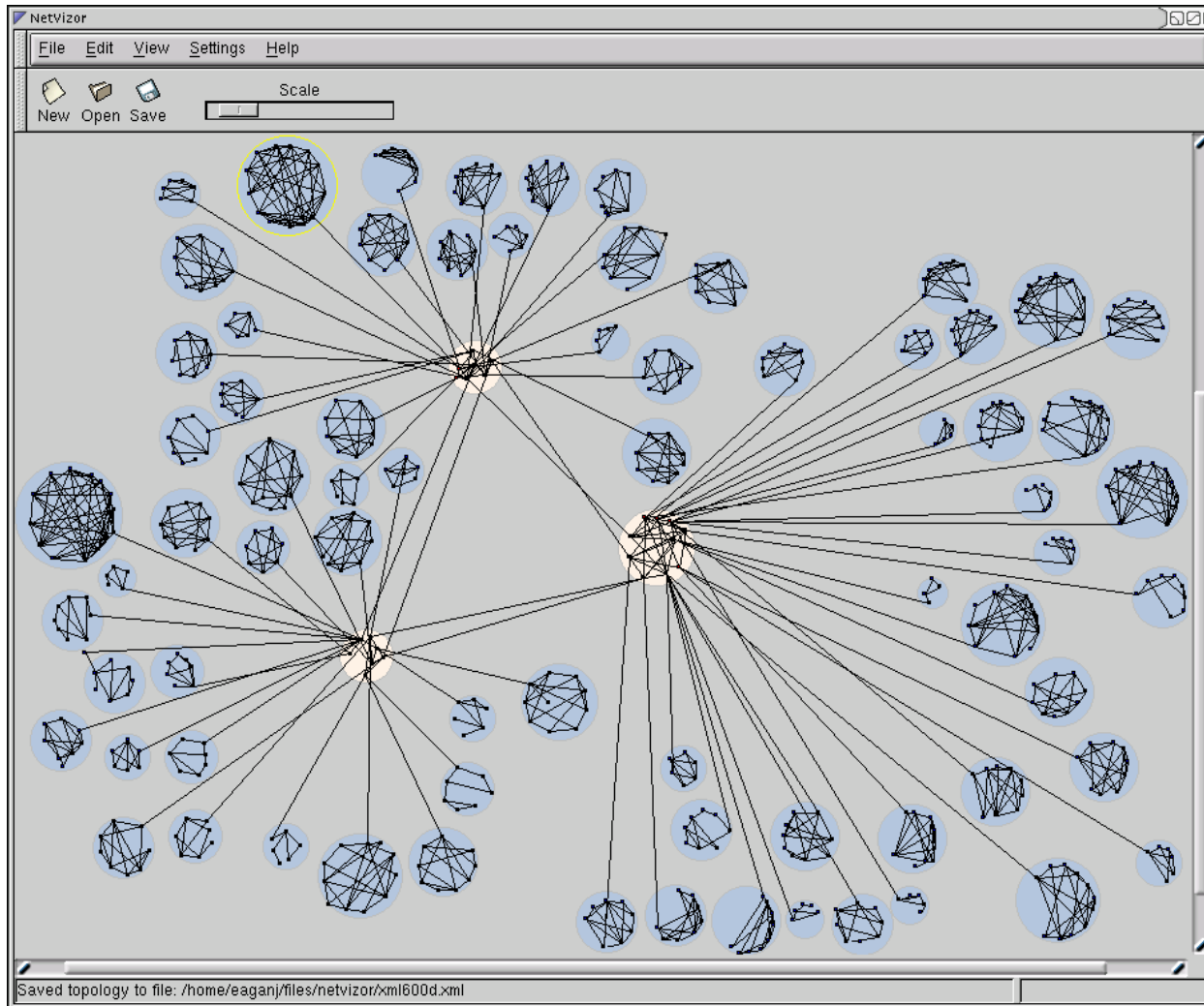
I Random graphs III: Waxman model



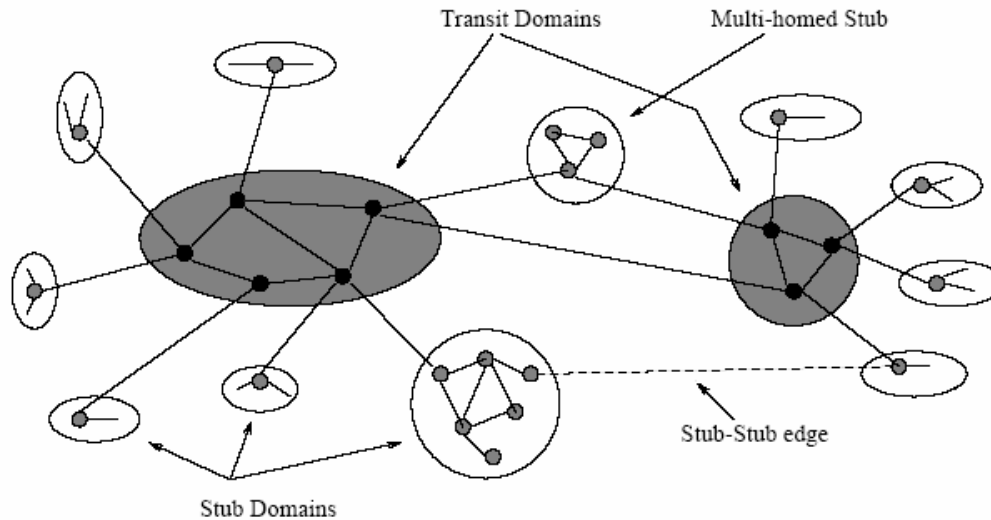
$$P(u, v) = a \exp\left(\frac{-d}{bL}\right)$$

- » **Caution: this is not an exponential distribution. It is a Bernoulli one (edge yes or no) where the probability of ‘edge’ depends on the Euclidean distance d of the new nodes u and v , and on the system areas diameter L .**
- » **Parameter a adjusts the decay w.r.t. distance, b the general level of ‘success’.**

I Hierarchical random graphs



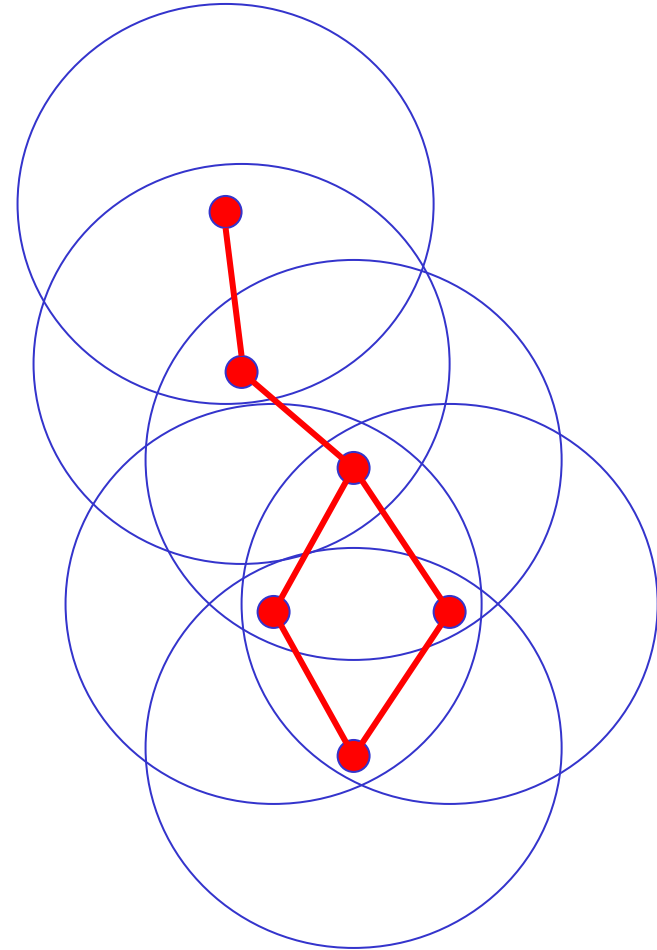
I Hierarchical random graphs



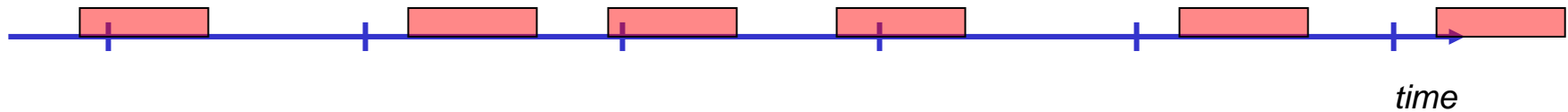
- » **Construct connected random graph: each node represents an entire transit domain**
- » **Each node is replaced by a connected random graph representing a transit network**
- » **For each node in transit networks: construct connected random graph representing the associated stub networks**
- » **Add some more edges between transit-and-stub and stub-and-stub networks.**

I Unit disc graph (for wireless scenarios)

- » For wireless scenarios – idealized ‘transmission ranges’
- » There is an edge between nodes u and v if distance between nodes is smaller than a given value t .



I Constant bit-rate traffic with jitter

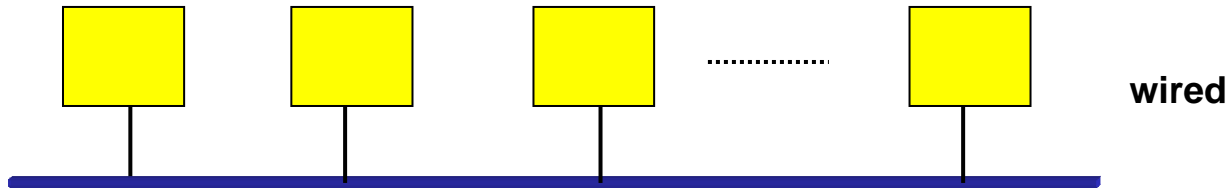


- » Constant bit rate traffic is parameterized via
 - Rate: the sending rate or
 - Interval: interval between packets and
 - PacketSize: the constant size of the packets generated
- » Introducing jitter: random flag indicating whether or not to introduce random “noise” in the scheduled departure times (default is off)
- » Example:
 - packetSize 48
 - rate 64Kb
 - random 1
- » Example (NS-2, /tools/cbr_traffic.cc):

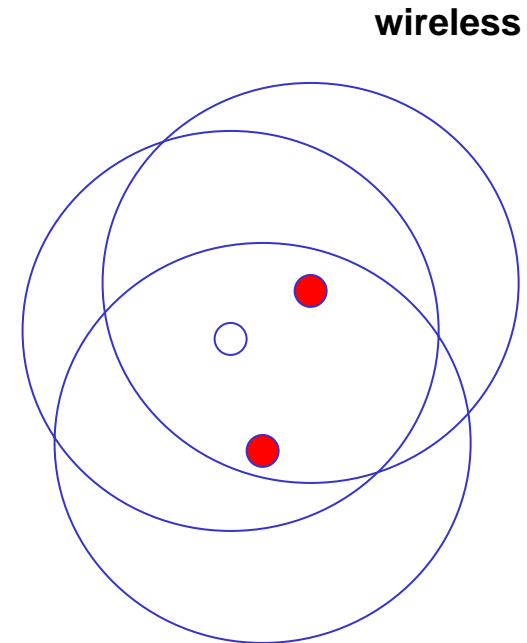
```
double t = interval_; if (random_)  
  
    t += interval_ * Random::uniform(-0.5, 0.5);
```

I Multiaccess communication: problem statement

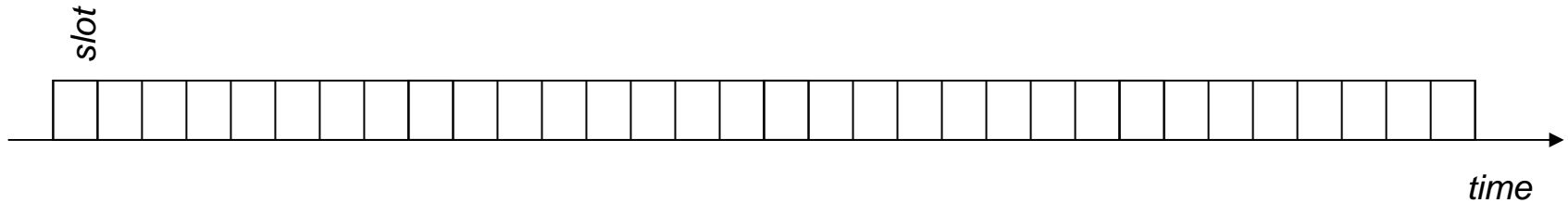
- » Randomness as a mechanism for self-organization



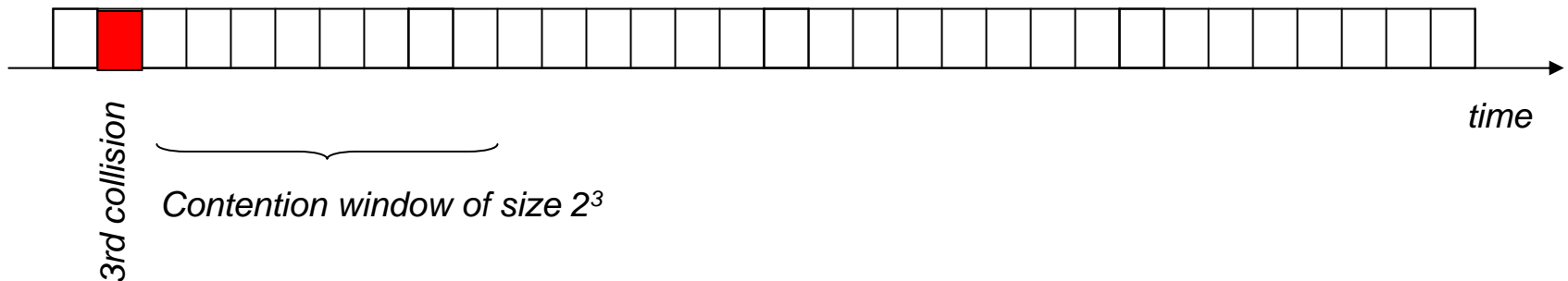
- » Assume: no centralized control
- » How to organize access to shared medium?
- » When two or more stations send a packet at the 'same' time, a collision occurs



I Multiaccess communication: exponential backoff



- » **Binary exponential backoff:**
- » **If a packet has been transmitted unsuccessfully i times, the transmission is rescheduled for a randomly chosen slot in the 'contention window' of size 2^i slots.**
- » **Self-adjustment to number of competing stations.**
- » **Example:**



I Multiaccess communication: 10Mbps Ethernet

Collision on attempt no.	Contention window	Backoff times
1	0 ... 1	0 ... 51.2 μ s
2	0 ... 3	0 ... 153.6 μ s
3	0 ... 7	0 ... 358.4 μ s
4	0 ... 15	0 ... 768 μ s
5	0 ... 31	0 ... 1.59 ms
6	0 ... 63	0 ... 3.23 ms
7	0 ... 127	0 ... 6.5 ms
8	0 ... 255	0 ... 13.1 ms
9	0 ... 511	0 ... 26.2 ms
10-15	0 ... 1023	0 ... 52.4 ms
16	N/A	Discard frame

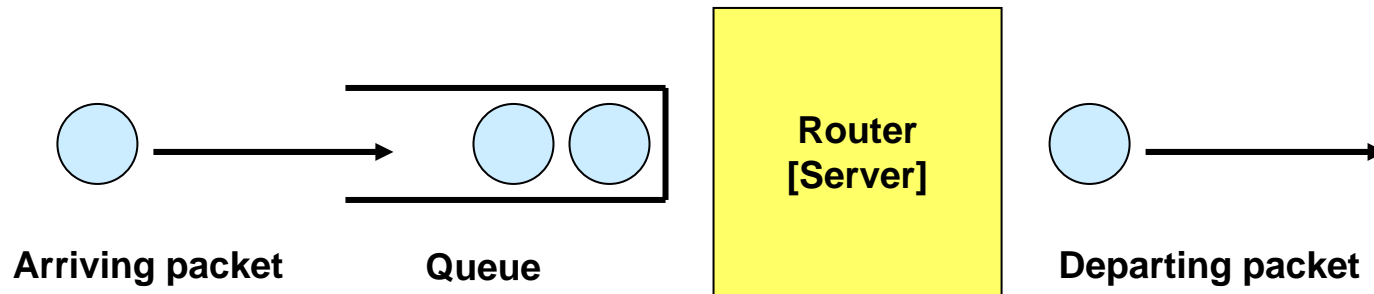
I Multiaccess communication: NS-2 example

```
rtime = (Random::random() % cw) * mac->phymib_.getSlotTime();
```

- » From ns-2.27/mac/mac-timers.cc
- » `random()` returns random integer in $[0, \text{MAXINT}]$
- » Efficient?
- » Other approach starting with $U(0,1)$:

$$\lfloor (cw + 1)U \rfloor$$

II Generating other random variates: introduction



How to generate exponentially distributed inter-arrival times?

- » Algorithms to produce observations (“variates”) from some desired input distribution (exponential, gamma, etc.)
- » Formal algorithm - depends on desired distribution.
- » But *all* algorithms have the same general form:
 - Generate one or more IID $U(0, 1)$ random numbers
 - Transformation (depends on desired distribution)
 - Return $X \sim$ desired distribution
- » Note critical importance of a good random-number generator

II Criteria

May be several algorithms for a desired input distribution form; want:

» **Exact:** X has *exactly* (not approximately) the desired distribution

– Example of approximate algorithm:

- Treat $Z = 6 - U_1 + U_2 + \dots + U_{12}$ as normal distribution $N(0,1)$
- Mean, variance correct; rely on Central Limit Theorem for *approximate* normality
- Range clearly incorrect

» **Efficient:**

- Low storage
- Fast (marginal, setup)
- Efficient regardless of parameter values (*robust*)

» **Simple:** Understand, implement (often tradeoff against efficiency)

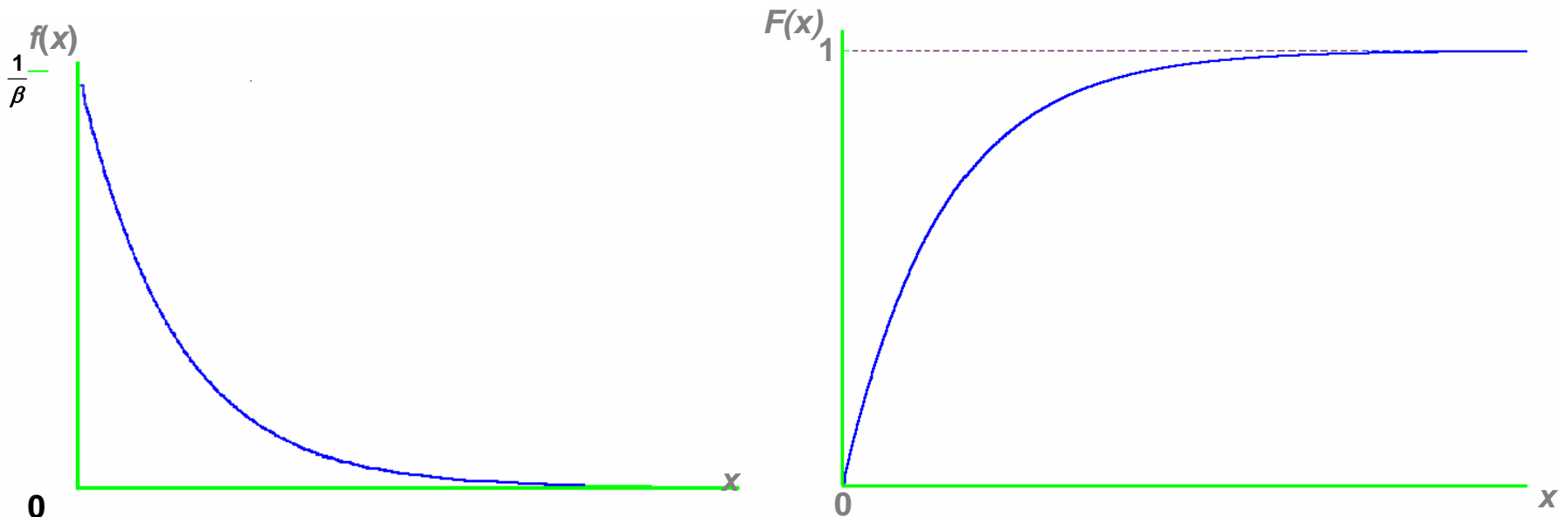
II Recap: distribution function

Definition:

the (cumulative) distribution function $F(x)$ for a random variable

$X: \Omega \rightarrow \mathbb{R}$ is defined as

$$F(x) = P(X \leq x) \quad \text{for} \quad -\infty < x < \infty$$



II Recap: strictly increasing function

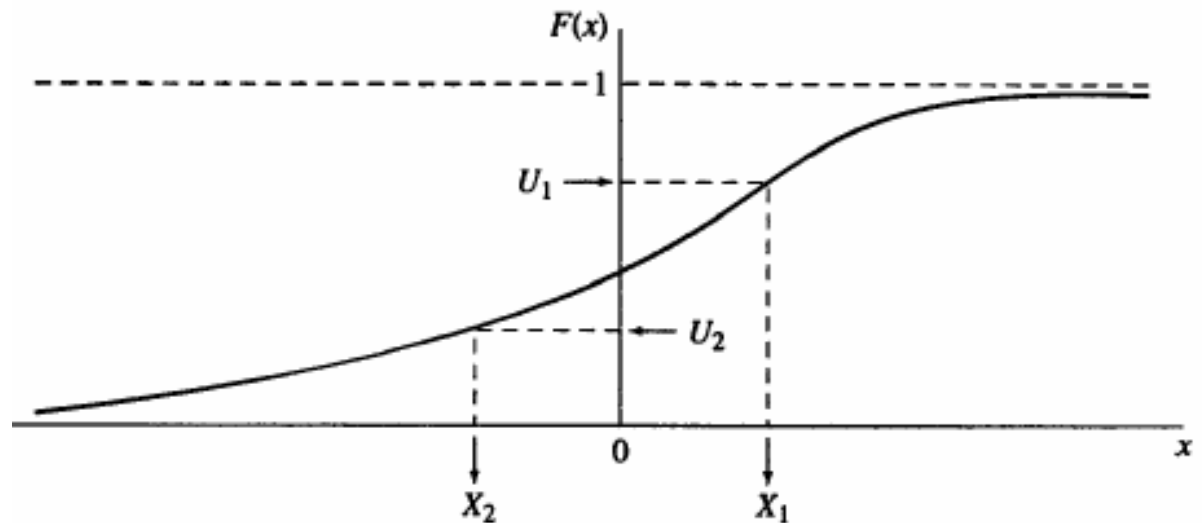
» A distribution function F is *strictly* increasing when

$$x_1 < x_2 \rightarrow F(x_1) < F(x_2)$$

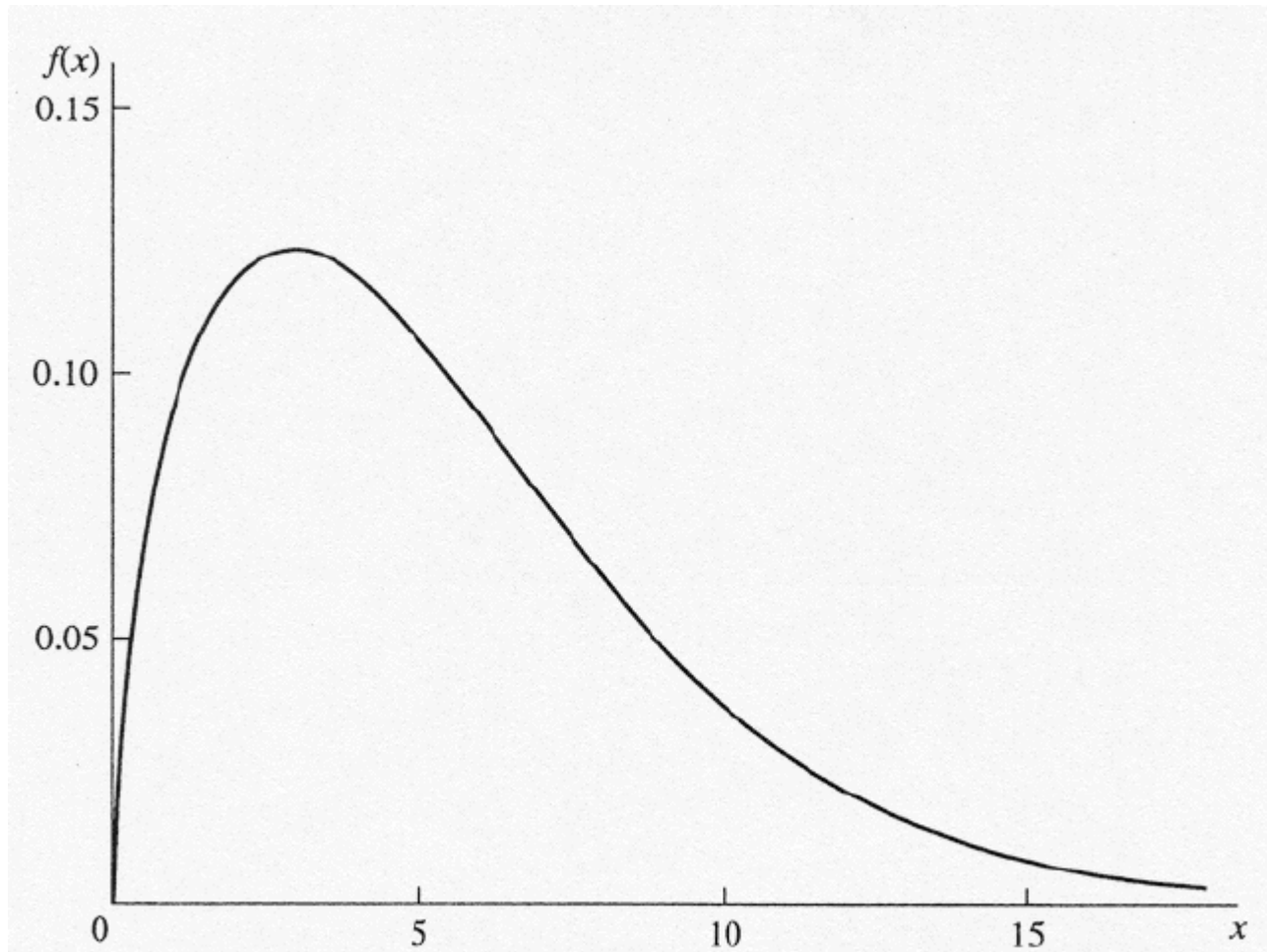
» For a distribution function F that is continuous and strictly increasing when $0 < F(x) < 1$, the inverse function F^{-1} exists on $(0,1)$.

II Inverse transform method

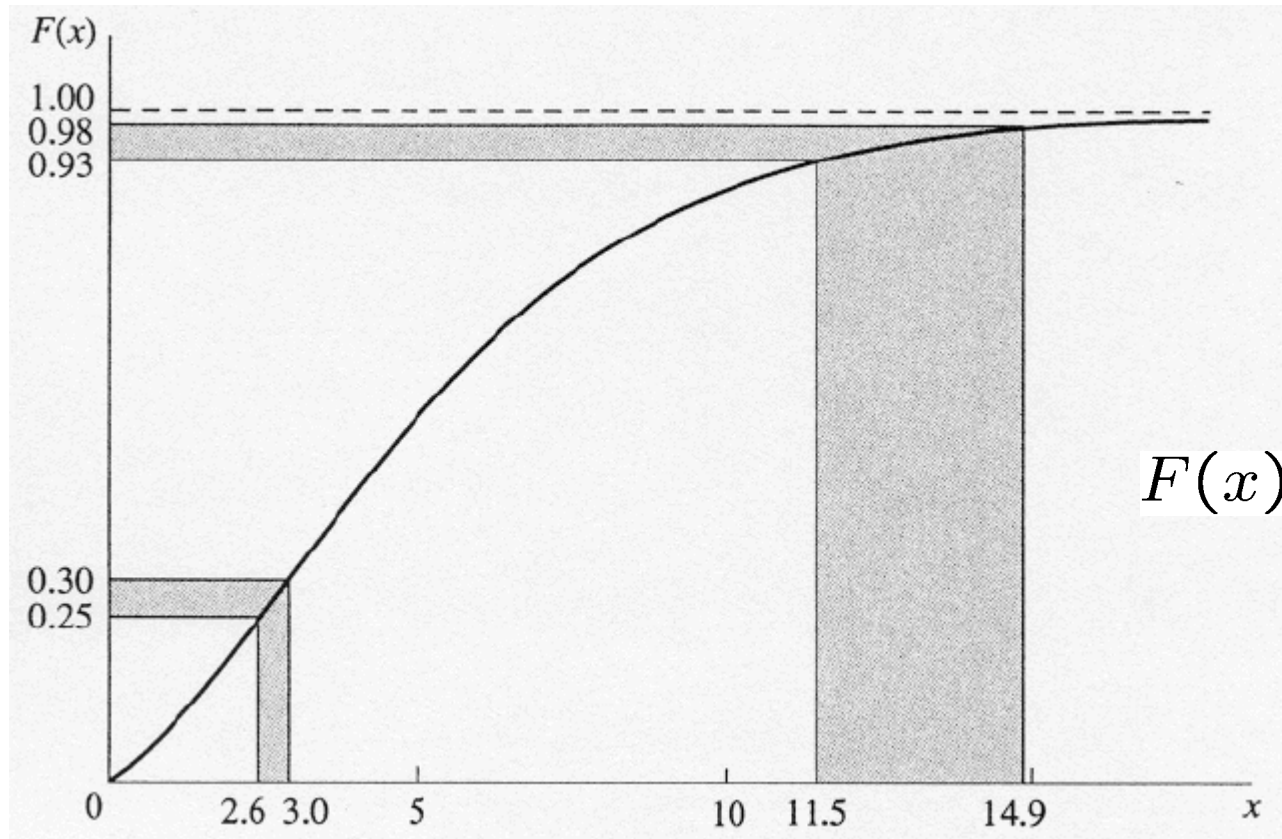
- » Goal: generate a random variate X that is continuous and has distribution function F that is continuous and strictly increasing when $0 < F(x) < 1$.
- » Strategy: Inverse transform algorithm
 1. Generate $U \sim U(0,1)$
 2. Return $X = F^{-1}(U)$



II Intuitive explanation I: density

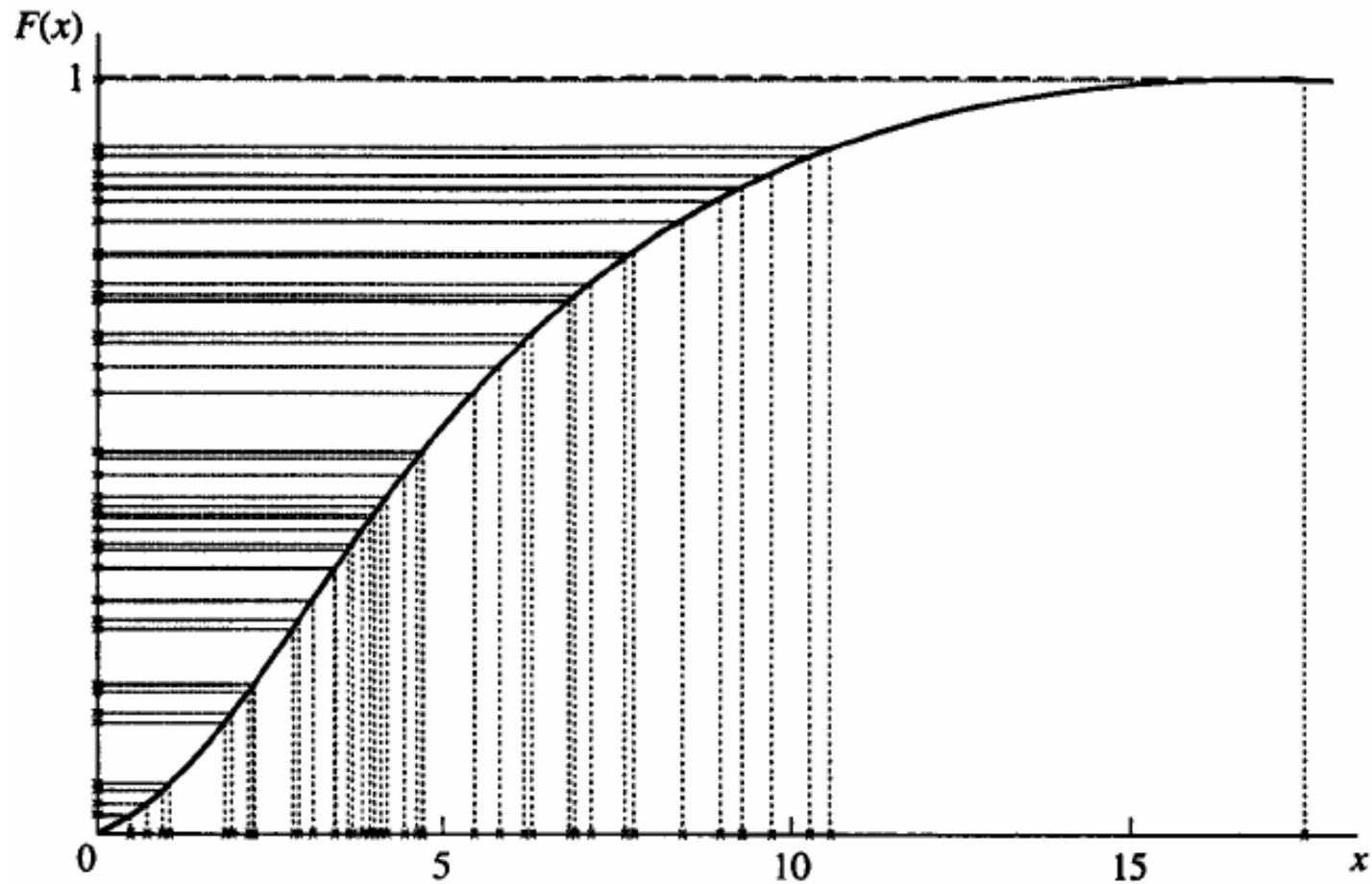


II Intuitive explanation II: distribution function



$$F(x) = P(X \leq x)$$

II Intuitive explanation III: algorithm at work (50 samples)



II Proof

$$P(X \leq x) = P(F^{-1}(U) \leq x)$$

» **Definition**

$$P(F^{-1}(U) \leq x) = P(U \leq F(x))$$

» **Properties of F**

$$P(U \leq F(x)) = F(x)$$

» **Uniform distribution,
definition of distribution
function**

$$P(X \leq x) = F(x)$$

II Example: exponential distribution

» **Density**

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

» **Distribution function**

$$F(x) = \begin{cases} 1 - e^{-x/\beta} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

» **Inverse transform**

$$F^{-1}(u) = -\beta \ln(1 - u)$$



uniformly distributed

II Exponential distribution: C code

From M/M/1 example:

```
float expon(float mean) /* Exponential variate
    generation function. */

{ /* Return an exponential random variate with mean
    "mean". */

    return -mean * log(lcgrand(1)); }
```

From NS-2 (tools/rng.h):

```
inline double exponential ()

    {return (-log(uniform()));}

inline double exponential (double r)

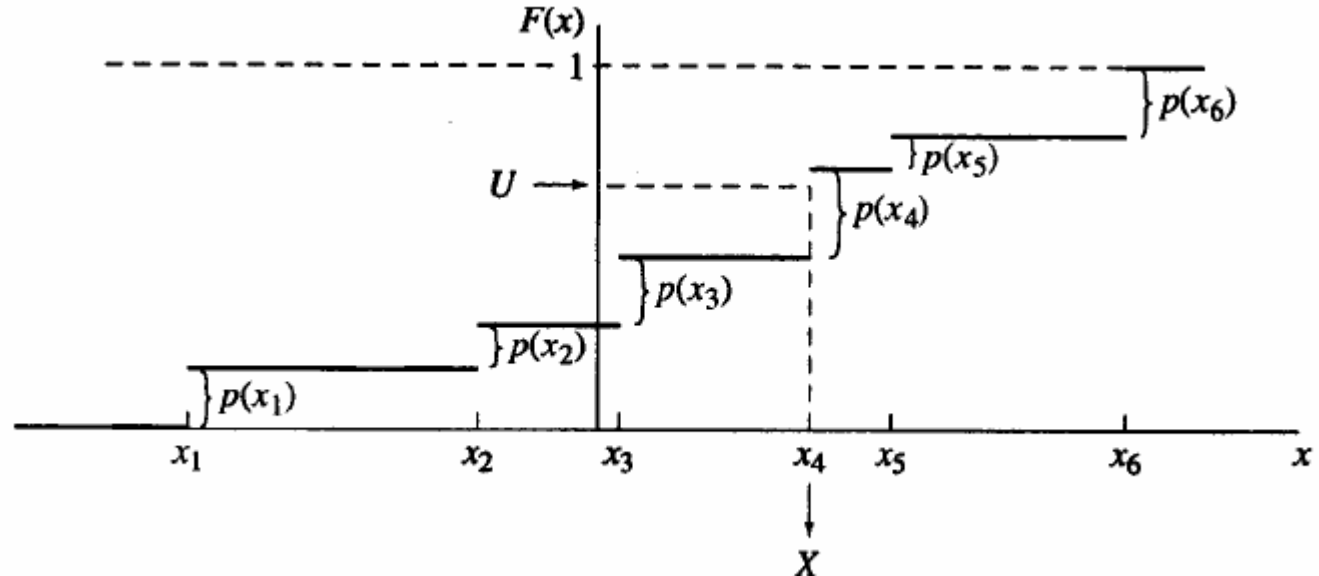
    {return (r * exponential());}
```

II Inverse transform method for discrete distributions

- » Assume: random variable X can take only values x_1, x_2, \dots with $x_1 < x_2 < \dots$
- » Probability mass function: $p(x_i) = P(X = x_i)$
- » Distribution function: $F(x) = P(x \leq X) = \sum_{x_i \leq x} p(x_i)$

Algorithm:

1. Generate $U \sim U(0,1)$
2. Determine the smallest positive integer I such that $U \leq F(x_i)$, and return X



II Discrete case: efficiency considerations

Step 2 involves a “search” of some kind; several computational options:

- Direct left-to-right search—if $p(x_i)$'s fairly constant
- If $p(x_i)$'s vary a lot, first sort into decreasing order, look at biggest one first, ..., smallest one last—more likely to terminate quickly
- Example:
Probabilities $\{p_1=1/8, p_2=1/8, p_3=1/4, p_4=1/2\}$.
A: order 1, 2, 3, 4. Expected number of comparisons: $11/8$
B: order 4, 3, 2, 1. Expected number: $9/8$
- Exploit special properties of the form of the $p(x_i)$'s, if possible
 - To facilitate computation of $F(x)$



II Generalizations and limits

» Generalized Inverse-Transform Method

- Valid for *any* CDF $F(x)$: return $X = \min\{x: F(x) \geq U\}$, where $U \sim U(0,1)$
 - Continuous, possibly with flat spots (i.e., not strictly increasing)
 - Discrete
 - Mixed continuous-discrete

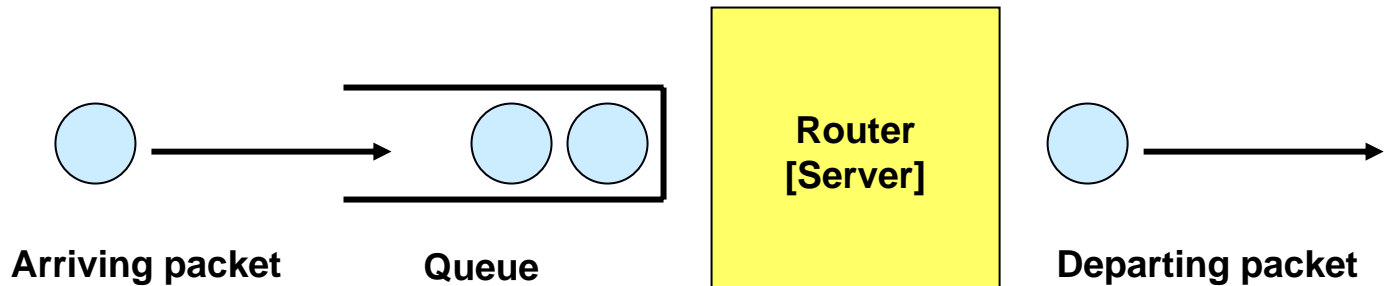
» Problems with Inverse-Transform Approach

- Must invert CDF, which may be difficult (numerical methods)
- May not be the fastest or simplest approach for a given distribution

III Modeling examples

» ... using exponential variates

» Recap: Inter-arrival times



» Examples for topologies

- Mobile case: random direction mobility

» Examples for synthetic data traffic

- On/off sources for generating bursty traffic

III Random direction mobility

- » Last lecture: random waypoint mobility
- » Various flavors around
- » Node chooses random direction sampled from uniform distribution over $[0, 2\pi)$
- » Node chooses speed either from uniform or normal distributions
- » Node changes direction/speed after a time interval sampled from an exponential distribution
- » At system area's boundary: wrap-up, reflection ...

III On/off sources for modeling bursty traffic

- » **NS-2: EXPOO_Traffic**—generates traffic according to an Exponential On/Off distribution. Packets are sent at a fixed rate during on periods, and no packets are sent during off periods. Both on and off periods are taken from an exponential. Packets are constant size.
- » **packetSize_** the constant size of the packets generated
- » **burst_time_** the average “on” time for the generator
- » **idle_time_** the average “off” time for the generator
- » **rate_** the sending rate during “on” times
- » **Example:**
 - **packetSize_ 210**
 - **burst_time_ 500ms**
 - **idle_time_ 500ms**
 - **rate_ 100k**

III VoIP modeling

- » **ITU-T recommendations for VoIP modeling:**
- » **On periods: exponential with mean 1.004s**
- » **Off periods: exponential with mean 1.587s**
- » **Sending rate during on periods: 88kbps**

Summary

- » **Examples of how to model network topologies and data traffic**
- » **Discussion on how to generate arbitrary random variates**
- » **Inverse transform method**
 - **Particularly good for exponential variates**

So far:

- **Generation of uniform, discrete and exponential variates**
- **Random topologies (static, wireless, mobile networks),
Synthetic data traffic (CBR, exponentially distributed
inter-arrival times, on/off-sources)**

References

» Exponential backoff

- Charles E. Spurgeon, *Ethernet – The Definite Guide*, O'Reilly, 2000; pp. 53-60
- Dimitri Bertsekas, Robert Gallager, *Data Networks*, 2nd ed., Prentice Hall, 1992, pp. 286

» Inverse transform method

- Averill M. Law, W. David Kelton, *Simulation Modeling and Analysis*, 3rd ed., McGrawHill, 2000

» Waxman model

- Bernard M Waxman Routing of multipoint connections *IEEE Journal on Selected Areas in Communications*, 6 (9): 1617-1622, 1988.

» Hierarchical random graphs

- Georgia Tech Internet Topology Map (GT-ITM), *models.ps*